PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 9, September 2016, Pages 3781–3795
http://dx.doi.org/10.1090/proc/13025
Article electronically published on May 6, 2016

ORTHOGONAL POLYNOMIALS FOR THE WEAKLY EQUILIBRIUM CANTOR SETS

GÖKALP ALPAN AND ALEXANDER GONCHAROV

(Communicated by Walter Van Assche)

Abstract. Let $K(\gamma)$ be the weakly equilibrium Cantor-type set introduced by the second author in an earlier work. It is proven that the monic orthogonal polynomials Q_{2^s} with respect to the equilibrium measure of $K(\gamma)$ coincide with the Chebyshev polynomials of the set. Procedures are suggested to find Q_n of all degrees and the corresponding Jacobi parameters. It is shown that the sequence of the Widom factors is bounded below.

1. Introduction

This paper is concerned with the spectral theory of orthogonal polynomials for measures supported on Cantor sets with a special emphasis on the purely singular continuous case. It should be noted that Cantor sets appear as supports of spectral measures for some important discrete Schrödinger operators used in physics (see e.g. the review [27] and [3]). We are interested in the following two problems related to orthogonal polynomials on Cantor-type sets. What can be said about the periodicity of corresponding Jacobi parameters? What is the notion of the Szegő class of measures on Cantor sets?

Concerning the first problem, the fundamental conjecture (see [21] and also Conjecture 3.1 in [18]) is that, for a large class of measures supported on Cantor sets, including the self-similar measures generated by linear iterated function systems (IFS), the corresponding Jacobi matrices are asymptotically almost periodic. Confirmation of this hypothesis may allow us to extend the methods used in [11, 12] for the finite gap sets to the Cantor sets with zero Lebesgue measure.

Concerning the second question, we mention that Szegö's theorem was generalized recently in [10] to the class of Parreau-Widom sets. Such sets may be of Cantor-type, but they must be of positive Lebesgue measure.

We can mention two main directions in the development of the theory of orthogonal polynomials for purely singular continuous measures. The first deals with a renormalization technique suggested by Mantica in [20], which enables us to efficiently compute Jacobi parameters (see e.g. [17, 18, 20]) for balanced measures via a linear IFS. Moreover, possible extensions of the notion of an isospectral torus for singular continuous measures can be found in [18, 22].

Received by the editors June 19, 2015 and, in revised form, October 22, 2015.
2010 *Mathematics Subject Classification.* Primary 42C05, 47B36; Secondary 31A15.
Key words and phrases. Orthogonal polynomials, equilibrium measure, Cantor sets, Jacobi matrices.
The authors were partially supported by a grant from Tübitak: 115F199.
The authors thank the anonymous referee for pointing out the articles [4, 8, 20–22].

On the other hand, there is a theory of orthogonal polynomials for equilibrium measures of real polynomial Julia sets (see e.g [4–7]). This includes simple formulas for orthogonal polynomials and recurrence coefficients, and almost periodicity of Jacobi matrices for certain Julia sets.

Here, we consider a family of Cantor sets $K(\gamma)$, introduced in [16]. A sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ serves as a parameter for the considered family of sets. By changing γ we can get sets of different logarithmic capacity and Hausdorff dimension. At least in known cases, the set $K(\gamma)$ is dimensional, that is, there exists a dimension function h such that for the corresponding Hausdorff measure Λ_h we have $0 < \Lambda_h(K(\gamma)) < \infty$. By [1], the equilibrium measure $\mu_{K(\gamma)}$ of $K(\gamma)$ and Λ_h are mutually absolutely continuous. This is not valid for geometrically symmetric zero Lebesgue measure Cantor sets, where, by [19] and followers, these measures are mutually singular.

We remark that the method of construction of the set is related to inverse polynomial image techniques. Thus, our results can be compared with [8, 15]. Furthermore, similarities between the results obtained here and for orthogonal polynomials on Julia sets are not mere coincidence. As soon as $\inf \gamma_k > 0$, $K(\gamma)$ can be considered as a generalized polynomial Julia set in the sense of Brück-Büger [9]. Moreover, some results of this paper can be transferred into a more general setting. For more details, we refer the reader to [2].

Our paper is organized as follows. In Section 2 we recall some facts from [16] about $K(\gamma)$ and show that the monic orthogonal polynomial Q_{2^s} of degree 2^s for $\mu_{K(\gamma)}$ coincides with the corresponding Chebyshev polynomial. In Sections 3 and 4 we suggest a procedure to find Q_n for $n \neq 2^s$. This allows us to analyze the asymptotic behavior of the Jacobi parameters $(a_n)_{n=1}^{\infty}$. Note that, if one can obtain a stronger version of Theorem 4.7 by showing that the limit of a_{j2^s+n} holds uniformly in n and j as in [6], this would imply that the Jacobi matrices considered here are almost periodic provided that $\sup \gamma_k \leq 1/6$.

Since $\mathrm{Cap}(K(\gamma))$ is known, we estimate (Section 5) the *Widom factors* $W_n := \frac{a_1 \cdots a_n}{\mathrm{Cap}(K(\gamma))^n}$ and check the Widom condition that characterizes the Szegő class of Jacobi matrices in the finite gap case. In the last section we discuss a possible version of the Szegő condition for singular continuous measures. At least for $\gamma_s \leq 1/6, s \in \mathbb{N}$, the Lebesgue measure of the set $K(\gamma)$ is zero, so it is not a Parreau-Widom set and Theorem 2 of [10] cannot be applied.

For the basic concepts of the theory of logarithmic potential, see e.g [25], log denotes the natural logarithm, $\mathrm{Cap}(\cdot)$ stands for the logarithmic capacity, $0^0 := 1$. We denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

2. Orthogonal polynomials

Given a sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$ define $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$. Let

$$P_1(x) := x - 1 \quad \text{and} \quad P_{2^{s+1}}(x) := P_{2^s}(x) \cdot (P_{2^s}(x) + r_s) \tag{2.1}$$

for $s \in \mathbb{N}_0$ in a recursive fashion. Thus, $P_2(x) = x \cdot (x - 1)$ for each γ, whereas, for $s \geq 2$, the polynomial P_{2^s} essentially depends on the parameter γ. For $s \in \mathbb{N}_0$

consider a nested sequence of sets

$$E_s = \{x \in \mathbb{R} : P_{2^{s+1}}(x) \le 0\} = \left(\frac{2}{r_s} P_{2^s} + 1\right)^{-1}([-1,1]) = \bigcup_{j=1}^{2^s} I_{j,s},$$

where $I_{j,s}$ are closed *basic* intervals of the $s-$th level which are necessarily disjoint. Let $l_{j,s}$ stand for the length of $I_{j,s}$ where we enumerate them from the left to the right. By Lemma 2 in [16], $\max_{1\le j\le 2^s} l_{j,s} \to 0$ as $s \to \infty$. Therefore, $K(\gamma) := \bigcap_{s=0}^{\infty} E_s$ is a Cantor set.

By Lemma 6 in [16],

$$\gamma_1 \cdots \gamma_s < l_{i,s} < \exp\left(16 \sum_{k=1}^{s} \gamma_k\right) \gamma_1 \cdots \gamma_s, \quad 1 \le i \le 2^s,$$

provided $\gamma_k \le 1/32$ for all k. Then the Lebesgue measure $|E_s|$ of the set E_s does not exceed $(\sqrt{e}/16)^s$. Hence $|K(\gamma)| = 0$ for such a γ. In Section 4 we will show that $|K(\gamma)| = 0$ as well if $\gamma_k \le 1/6$ for all k.

On the other hand, by choosing $(\gamma_k)_{k=1}^{\infty}$ sufficiently close to $1/4$, we can obtain Cantor sets with positive Lebesgue measure. What is more, in the limit case, when all $\gamma_k = 1/4$, we get $E_s = [0,1]$ for all s and $K(\gamma) = [0,1]$ (see Example 1 in [16]).

In addition, using the Green function $g_{\mathbb{C}\setminus K(\gamma)}$ (see Corollary 1 and Section 6 in [16]), one can easily find $\mathrm{Cap}(K(\gamma)) = \exp\left(\sum_{k=1}^{\infty} 2^{-k} \log \gamma_k\right)$. In the paper we assume $\mathrm{Cap}(K(\gamma)) > 0$. Let $\mu_{K(\gamma)}$ denote the equilibrium measure on the set, and $||\cdot||$ be the norm in the corresponding Hilbert space. From Corollary 3.2 in [1] we have $\mu_{K(\gamma)}(I_{j,s}) = 2^{-s}$ for all s and $1 \le j \le 2^s$, provided $\gamma_k \le 1/32$ for all k.

From now on, by Q_n we denote the monic orthogonal polynomial of degree $n \in \mathbb{N}$ with respect to $\mu_{K(\gamma)}$. The main result of this section is that, for $n = 2^s$ with $s \in \mathbb{N}_0$, the polynomial Q_n coincides with the corresponding Chebyshev polynomial for $K(\gamma)$. The next two theorems will play a crucial role.

Theorem 2.1 ([16], Prop.1)**.** *For each $s \in \mathbb{N}_0$ the polynomial $P_{2^s} + r_s/2$ is the Chebyshev polynomial for $K(\gamma)$.*

Remark 2.2. Only the values $s \in \mathbb{N}$ were considered in [16]. But, clearly, for $s = 0$ the polynomial $P_1(x) + 1/2 = x - 1/2$ is Chebyshev.

Remark 2.3. Since real polynomials are considered here and the alternating set for $P_{2^s} + r_s/2$ consists of $2^s + 1$ points, the Chebyshev property of this polynomial follows by the Chebyshev alternation theorem.

Theorem 2.4 ([26], III.T.3.6)**.** *Let $K \subset \mathbb{R}$ be a non-polar compact set. Then the normalized counting measures on the zeros of the Chebyshev polynomials converge to the equilibrium measure of K in the weak-star topology.*

For $s \in \mathbb{N}$, the polynomial $P_{2^s} + r_s\,/\,2$ has simple real zeros $(x_k)_{k=1}^{2^s}$ which are symmetric about $x = 1/2$. Let us denote by ν_s the normalized counting measure at these points, that is, $\nu_s = 2^{-s} \sum_{k=1}^{2^s} \delta_{x_k}$.

Lemma 2.5. *Let $s > m$ with $s, m \in \mathbb{N}_0$. Then $\int \left(P_{2^m} + \frac{r_m}{2}\right) d\nu_s = 0$.*

Proof. For $m = 0$ we have the result by symmetry. Suppose $m \ge 1$. By (2.1), at the points $(x_k)_{k=1}^{2^s}$, we have

$$P_{2^s} + \frac{r_s}{2} = (P_{2^{s-1}})^2 + r_{s-1} P_{2^{s-1}} + \frac{r_s}{2} = 0.$$

The discriminant of the equation is positive. Therefore, the roots satisfy

$$(P_{2^{s-1}} + \alpha_{s-1}^1)(P_{2^{s-1}} + \alpha_{s-1}^2) = 0,$$

where $\alpha_{s-1}^1 + \alpha_{s-1}^2 = r_{s-1}$ and $0 < \alpha_{s-1}^1, \alpha_{s-1}^2 < r_{s-1}$. Thus, a half of the points satisfies $P_{2^{s-1}} + \alpha_{s-1}^1 = 0$ while the other half satisfies $P_{2^{s-1}} + \alpha_{s-1}^2 = 0$.

Rewriting the equation $P_{2^{s-1}} + \alpha_{s-1}^1 = 0$, we see that

$$P_{2^{s-2}}^2 + r_{s-2} P_{2^{s-2}} + \alpha_{s-1}^1 = 0.$$

Since $r_{s-2}^2 > 4r_{s-1} > 4\alpha_{s-1}^1$, this yields

$$(P_{2^{s-2}} + \alpha_{s-2}^1)(P_{2^{s-2}} + \alpha_{s-2}^2) = 0$$

with $\alpha_{s-2}^1 + \alpha_{s-2}^2 = r_{s-2}$ and $0 < \alpha_{s-2}^1, \alpha_{s-2}^2 < r_{s-2}$. By the same argument, the second half of the roots satisfies

$$(P_{2^{s-2}} + \alpha_{s-2}^3)(P_{2^{s-2}} + \alpha_{s-2}^4) = 0$$

with $\alpha_{s-2}^3 + \alpha_{s-2}^4 = r_{s-2}$ and $0 < \alpha_{s-2}^3, \alpha_{s-2}^4 < r_{s-2}$.

Since at each step $r_{i-1}^2 > 4r_i$ we can continue this procedure until obtaining $P_{2^{m+1}}$. So we can decompose the Chebyshev nodes $(x_k)_{k=1}^{2^s}$ into 2^{s-m-1} groups. All 2^{m+1} nodes from the $i-$th group G_i satisfy

$$P_{2^{m+1}} + \alpha_{m+1}^i = 0, \;\; 0 < \alpha_{m+1}^i < r_{m+1}.$$

By using these 2^{s-m-1} equations we finally obtain

$$(P_{2^m} + \alpha_m^{2i-1})(P_{2^m} + \alpha_m^{2i}) = 0$$

where $\alpha_m^{2i-1} + \alpha_m^{2i} = r_m$. Thus, given fixed i with $1 \le i \le 2^{s-m-1}$, for 2^m points from the group G_i we have $P_{2^m} = -\alpha_m^{2i-1}$, whereas for the other half, $P_{2^m} = -\alpha_m^{2i}$. Consequently,

$$\int \left(P_{2^m} + \frac{r_m}{2}\right) d\nu_s = \int P_{2^m} d\nu_s + \frac{r_m}{2} = \frac{\sum_{i=1}^{2^{s-m-1}} 2^m(-\alpha_m^{2i-1} - \alpha_m^{2i})}{2^s} + \frac{r_m}{2} = 0.$$

□

Lemma 2.6. *Let* $0 \le i_1 < i_2 < \dots i_n < s$. *Then*

(a) $\displaystyle\int P_{2^{i_1}} P_{2^{i_2}} \cdots P_{2^{i_n}} d\nu_s = \int P_{2^{i_1}} d\nu_s \int P_{2^{i_2}} d\nu_s \cdots \int P_{2^{i_n}} d\nu_s = (-1)^n \prod_{k=1}^n \frac{r_{i_k}}{2}.$

(b) $\displaystyle\int \left(P_{2^{i_1}} + \frac{r_{i_1}}{2}\right)\left(P_{2^{i_2}} + \frac{r_{i_2}}{2}\right) \cdots \left(P_{2^{i_n}} + \frac{r_{i_n}}{2}\right) d\nu_s = 0.$

Proof. (a) Suppose that $i_1 \ge 1$. As above, we can decompose the nodes $(x_k)_{k=1}^{2^s}$ into 2^{s-i_1-1} equal groups such that the nodes from the $j-$th group satisfy an equation

$$(P_{2^{i_1}} + \alpha_{i_1}^{2j-1})(P_{2^{i_1}} + \alpha_{i_1}^{2j}) = 0$$

with $\alpha_{i_1}^{2j-1} + \alpha_{i_1}^{2j} = r_{i_1}$. If, on some set, $(P_{2^k} + \alpha)(P_{2^k} + \beta) = 0$ with $\alpha + \beta = r_k$, then $P_{2^{k+1}} = P_{2^k}^2 + P_{2^k} r_k = -\alpha\beta$. Hence, for each $i \in \mathbb{N}$, the polynomial $P_{2^{k+i}}$ is constant on this set. Therefore the function $P_{2^{i_2}} \dots P_{2^{i_n}}$ takes the same value for all x_k from the $j-$th group. This allows us to apply the argument of Lemma 2.5:

$$\int P_{2^{i_1}} P_{2^{i_2}} \cdots P_{2^{i_n}} d\nu_s = -\frac{r_{i_1}}{2} \int P_{2^{i_2}} P_{2^{i_3}} \cdots P_{2^{i_n}} d\nu_s.$$

This equality is valid also for $i_1 = 0$ since

$$\int \left(P_1 + \frac{1}{2}\right) P_{2^{i_2}} \cdots P_{2^{i_n}} \, d\nu_s = 0,$$

by symmetry. Proceeding this way, the result follows, since $-r_m/2 = \int P_{2^m} d\nu_s$, by Lemma 2.5.

(b) Opening the parentheses yields

$$\int P_{2^{i_1}} P_{2^{i_2}} \cdots P_{2^{i_n}} \, d\nu_s + \sum_{k=1}^{n} \frac{r_{i_k}}{2} \int \prod_{j \neq k} P_{2^{i_j}} \, d\nu_s + \cdots + \prod_{k=1}^{n} \frac{r_{i_k}}{2}.$$

By Lemma 2.5 and part (a), this is

$$\prod_{k=1}^{n} \frac{r_{i_k}}{2} \cdot \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0.$$

□

Remark 2.7. We can use $\mu_{K(\gamma)}$ instead of ν_s in Lemma 2.5 and Lemma 2.6 since, by Theorem 2.4, $\nu_s \to \mu_{K(\gamma)}$ in the weak-star topology.

Theorem 2.8. *The monic orthogonal polynomial Q_{2^s} with respect to the equilibrium measure $\mu_{K(\gamma)}$ coincides with the corresponding Chebyshev polynomial $P_{2^s} + r_s/2$ for all $s \in \mathbb{N}_0$.*

Proof. For $s = 0$ we have the result by symmetry. Let $s \geq 1$. Each polynomial $P(x)$ of degree less than 2^s is a linear combination of polynomials of the type

$$\left(P_{2^{s-1}}(x) + \frac{r_{s-1}}{2}\right)^{n_{s-1}} \cdots \left(P_2(x) + \frac{r_1}{2}\right)^{n_1} \left(x - \frac{1}{2}\right)^{n_0}$$

with $n_i \in \{0, 1\}$. By Lemma 2.6, $P_{2^s} + r_s \,/\, 2$ is orthogonal to all polynomials of degree less than 2^s, so it is Q_{2^s}. □

By (2.1), we immediately have

Corollary 2.9. $Q_{2^{s+1}} = Q_{2^s}^2 - (1 - 2\,\gamma_{s+1})\, r_s^2/4$ *for* $s \in \mathbb{N}_0$.

3. Some products of orthogonal polynomials

So far we only obtain orthogonal polynomials of degree 2^s. We try to find Q_n for other degrees. By Corollary 2.9, since $\int Q_{2^{s+1}} d\mu_{K(\gamma)} = 0$, we have

$$\|Q_{2^s}\|^2 = \int Q_{2^s}^2 d\mu_{K(\gamma)} = (1 - 2\,\gamma_{s+1})\, r_s^2/4 \tag{3.1}$$

and

$$Q_{2^{s+1}} = Q_{2^s}^2 - \|Q_{2^s}\|^2, \ \forall s \in \mathbb{N}_0. \tag{3.2}$$

Our next goal is to evaluate $\int A \, d\mu_{K(\gamma)}$ for A-*polynomial* of the form

$$A = (Q_{2^{s_n}})^{i_n} (Q_{2^{s_{n-1}}})^{i_{n-1}} \cdots (Q_{2^{s_1}})^{i_1}, \tag{3.3}$$

where $s_n > s_{n-1} > \ldots > s_1 > 0$ and $i_1, i_2, \ldots, i_n \in \{1, 2\}$.

The next lemma is basically a consequence of (3.2).

Lemma 3.1. *Let A be a polynomial satisfying* (3.3). *Then the following propositions hold:*

(a) *If $i_n = 2$, then* $\int A\, d\mu_{K(\gamma)} = \|Q_{2^{s_n}}\|^2 \int Q_{2^{s_{n-1}}}^{i_{n-1}} \cdots Q_{2^{s_1}}^{i_1}\, d\mu_{K(\gamma)}$.

(b) *Suppose that $n = k + m$ with $i_n = i_{n-1} = \ldots = i_{k+1} = 1$ and $i_k = 2$. In addition, let $s_{k+j} = s_k + j$ for $1 \le j \le m$. Then*
$$\int A\, d\mu_{K(\gamma)} = \|Q_{2^{s_n}}\|^2 \int Q_{2^{s_{k-1}}}^{i_{k-1}} \cdots Q_{2^{s_1}}^{i_1}\, d\mu_{K(\gamma)}.$$

(c) *If $i_k = 1$ and $s_k \ge s_{k-1}+2$ for some $k \in \{2, 3, \ldots, n\}$, then* $\int A\, d\mu_{K(\gamma)} = 0$.

(d) *If $i_1 = 1$, then* $\int A\, d\mu_{K(\gamma)} = 0$.

Proof. (a) Using (3.2), we have $Q_{2^{s_n}}^2 = Q_{2^{s_n+1}} + \|Q_{2^{s_n}}\|^2$. The result easily follows since the degree of $Q_{2^{s_{n-1}}}^{i_{n-1}} \cdots Q_{2^{s_1}}^{i_1}$ is less than 2^{s_n+1}.

(b) Here $A = Q_{2^{s_n}}\, Q_{2^{s_{n-1}}} \cdots Q_{2^{s_{k+1}}} Q_{2^{s_k}}^2 \cdot P$ with $P = Q_{2^{s_{k-1}}}^{i_{k-1}} \cdots Q_{2^{s_1}}^{i_1}$. Observe that $\deg P < 2^{s_{k-1}+2} \le 2^{s_{k+1}}$. We apply (3.2) repeatedly. First, since $s_{k+1} = s_k + 1$, we have $Q_{2^k}^2 = Q_{2^{s_{k+1}}} + ||Q_{2^{s_k}}||^2$. Similarly, $Q_{2^{s_{k+1}}} Q_{2^{s_k}}^2 = Q_{2^{s_k+2}} + ||Q_{2^{s_{k+1}}}||^2 + Q_{2^{s_{k+1}}}\, ||Q_{2^{s_k}}||^2$. After m steps we write A in the form $(Q_{2^{s_n+1}} + \|Q_{2^{s_n}}\|^2 + \mathcal{L})\, P$, where $\mathcal{L}$ is a linear combination of the polynomials $Q_{2^{s_n}},\ Q_{2^{s_n}} Q_{2^{s_{n-1}}}, \cdots, Q_{2^{s_n}} Q_{2^{s_{n-1}}} \cdots Q_{2^{s_{k+1}}}$. Here, $2^{s_n} > 2^{s_{n-1}} + \cdots + 2^{s_{k+1}} + \deg P$. By orthogonality, all terms vanish after integration, except $\|Q_{2^{s_n}}\|^2\, P$, which is the desired conclusion.

(c) Let us take the maximal k with such property. Repeated application of (a) and (b) enables us to reduce $\int A\, d\mu_{K(\gamma)}$ to $C \int A_1\, d\mu_{K(\gamma)}$ with $C > 0$ and $A_1 = Q_{2^{s_m}} \cdots Q_{2^{s_k}} \cdot R$, where $R = Q_{2^{s_{k-1}}}^{i_{k-1}} \cdots Q_{2^{s_1}}^{i_1}$ with $\deg R < 2^{s_{k-1}+2} \le 2^{s_k}$. Comparing the degrees gives the result.

(d) Take the largest k with $i_1 = i_2 = \cdots = i_k = 1$. Then, as above, $\int A\, d\mu_{K(\gamma)} = C \cdot \int Q_{2^{s_k}} \cdots Q_{2^{s_1}}\, d\mu_{K(\gamma)} = 0$, since the degree of the first polynomial exceeds the common degree of others. □

Theorem 3.2. *For $A-$polynomial given in* (3.3), *let $c_k = (i_k - 1)^{s_k - s_{k-1} - 1}$ and $c = \prod_{k=1}^n c_k$. Here, $s_0 := -1$ and $i_{n+1} := 2$. Then $\int A\, d\mu_{K(\gamma)} = c \cdot \prod_{k=1}^n ||Q_{2^k}||^{2(i_{k+1}-1)}$.*

Proof. First we remark that $c \in \{0, 1\}$. Clearly, $c_1 = (i_1 - 1)^{s_1} = 0$ if and only if $i_1 = 1$. For $k > 1$ we get $c_k = 0$ if and only if $i_k = 1$ and $s_k > s_{k-1} + 1$. Therefore, $c = 0$ just in the cases (c) and (d) above.

Let us show that the procedures $(a)-(d)$ of Lemma 3.1 allow us to find $\int A\, d\mu_{K(\gamma)}$ for all values of $(i_k)_{k=1}^n$ and $(s_k)_{k=1}^n$ stated after (3.3). Consider the string $\mathcal{I} = \{i_n, i_{n-1}, \cdots, i_1\}$. If $i_1 = 1$, then $c = 0$ and $\int A\, d\mu_{K(\gamma)} = 0$, by (d), so the result follows. Suppose $i_1 = 2$. Then we can decompose $\mathcal{I}$ into substrings of the types $\{2\}$, $\{1, 2\}, \cdots, \{1, \cdots, 1, 2\}$. The number and the ordering of such substrings may be arbitrary. We go over substrings of $\mathcal{I}$ in left-to-right order. If we meet $\{i_k\}$ with $i_k = 2$, then we use (a). Observe that here $i_{k+1} = 2$. Hence this substring contributes a term $||Q_{2^k}||^2$ into the product representing $\int A\, d\mu_{K(\gamma)}$. For a general substring $\{i_k, \cdots, i_{k-m}\}$ with $i_k = \cdots = i_{k-m+1} = 1$, $i_{k-m} = 2$ we also have $i_{k+1} = 2$. Consider the corresponding values s_j for $k - m \le j \le k$. Suppose that these numbers are consecutive, that is, $s_{j+1} = s_j + 1$ for $k - m \le j \le k-1$. Then we use the procedure (b). In this case, $i_{k+1} - 1 = 1$ and $i_{j+1} - 1 = 0$ for $k - m \le j \le k-1$.

As above, the substring gives a contribution $||Q_{2^k}||^2$ into the common product. Otherwise, $s_{j+1} \geq s_j + 2$ for some j. Then, by (c), $\int A\, d\mu_{K(\gamma)} = 0$. On the other hand, here, $c = c_j = 0$, so the desired representation for $\int A\, d\mu_{K(\gamma)}$ is valid as well. □

Corollary 3.3. *For $A-$polynomial given in* (3.3)*, let $A = A_1 \cdot Q_{2^{s_1}}^{i_1}$, so A_1 contains all terms of A except the last. Suppose $i_1 = i_2 = 2$. Then $\int A\, d\mu_{K(\gamma)} = ||Q_{2^{s_1}}||^2 \int A_1\, d\mu_{K(\gamma)}$.*

We will represent Q_n in terms of *B-polynomials* that are defined, for $2^m \leq n < 2^{m+1}$ with $m \in \mathbb{N}_0$, as

$$B_n = (Q_{2^m})^{i_m} (Q_{2^{m-1}})^{i_{m-1}} \dots (Q_1)^{i_1},$$

where $i_k \in \{0, 1\}$ is the $k-$th coefficient in the binary representation $n = i_m\, 2^m + \dots + i_0$.

Thus, B_n is a monic polynomial of degree n. The polynomials $B_{(2k+1)\cdot 2^s}$ and $B_{(2j+1)\cdot 2^m}$ are orthogonal for all $j, k, m, s \in \mathbb{N}_0$ with $s \neq m$. Indeed, if $\min\{m, s\} = 0$, then $\int B_{(2k+1)\cdot 2^s}\, B_{(2j+1)\cdot 2^m}\, d\mu_{K(\gamma)} = 0$, since one polynomial is symmetric about $x = 1/2$, whereas another is antisymmetric. Otherwise we use Lemma 3.1 (d). By (a), we have

$$||B_n||^2 = \prod_{k=0}^{m} ||Q_{2^k}||^{2i_k} = \prod_{k=0, i_k \neq 0}^{m} ||Q_{2^k}||^2.$$

Theorem 3.4. *For each $n \in \mathbb{N}$, let $n = 2^s(2k+1)$. The polynomial Q_n has a unique representation as a linear combination of $B_{2^s}, B_{3\cdot 2^s}, \dots, B_{(2k-1)\cdot 2^s}, B_{(2k+1)\cdot 2^s}$.*

Proof. Consider $P = a_0 B_{2^s} + a_1 B_{3\cdot 2^s} + \dots + a_{k-1} B_{(2k-1)\cdot 2^s} + B_{(2k+1)\cdot 2^s}$, where $(a_j)_{j=0}^{k-1}$ are chosen such that P is orthogonal to all $B_{(2j+1)2^s}$ with $j = 0, 1, \dots, k-1$. This gives a system of k linear equations with k unknowns $(a_j)_{j=0}^{k-1}$. The determinant of this system is the Gram determinant of linearly independent functions $(B_{(2j+1)2^s})_{j=0}^{k-1}$. Therefore it is positive and the system has a unique solution. In addition, as was remarked above, P is orthogonal to all $B_{(2j+1)\cdot 2^m}$ with $m \neq s$. Thus, P is a monic polynomial of degree n that is orthogonal to all polynomials of degree $< n$, so $P = Q_n$. □

Corollary 3.5. *The polynomial $Q_{2^s(2k+1)}$ is a linear combination of products of the type $Q_{2^{s_m}}\, Q_{2^{s_{m-1}}} \cdots Q_{2^s}$, so the smallest degree of $Q_{2^{s_j}}$ in every product is 2^s.*

To illustrate the theorem, we consider, for given $s \in \mathbb{N}_0$, the easiest cases with $k \leq 2$. Clearly, $Q_{2^s} = B_{2^s}$. Since $B_{3\cdot 2^s} = Q_{2^s} Q_{2^{s+1}}$, we take $Q_{3\cdot 2^s} = a_0 Q_{2^s} + Q_{2^{s+1}} Q_{2^s}$, where a_0 is such that $\int Q_{3\cdot 2^s}\, Q_{2^s}\, d\mu_{K(\gamma)} = 0$. By Lemma 3.1,

$$Q_{3\cdot 2^s} = Q_{2^{s+1}} Q_{2^s} - \frac{||Q_{2^{s+1}}||^2}{||Q_{2^s}||^2} Q_{2^s}.$$

Similarly, $B_{5\cdot 2^s} = Q_{2^s} Q_{2^{s+2}}$ and $Q_{5\cdot 2^s} = a_0 Q_{2^s} + a_1 Q_{2^{s+1}} Q_{2^s} + Q_{2^s} Q_{2^{s+2}}$ with

$$a_0 = \frac{||Q_{2^{s+2}}||^2}{||Q_{2^s}||^4 - ||Q_{2^{s+1}}||^2}, \quad a_1 = -a_0\, \frac{||Q_{2^s}||^2}{||Q_{2^{s+1}}||^2}.$$

Using (3.1), all coefficients can be expressed only in terms of $(\gamma_k)_{k=1}^{\infty}$. As k gets larger, the complexity of calculations increases.

Remark 3.6. In general, the polynomial Q_n is not Chebyshev. For example, $Q_3 = Q_1(Q_2 + a_0)$ with $a_0 = -\frac{(1-2\gamma_2)\gamma_1^2}{1-2\gamma_1}$. At least for small γ_1, the polynomial $Q_3(x) = (x-1/2)(x^2 - x + \gamma_1/2 + a_0)$ increases on the first basic interval $I_{1,1} = [0, l_{1,1}]$. Here, $l_{1,1}$ is the first solution of $P_2 = -r_1$, so $l_{1,1} = (1-\sqrt{1-4\gamma_1})/2$. If Q_3 is the Chebyshev polynomial, then, by the Chebyshev alternation theorem, $Q_3(l_{1,1}) = Q_3(1)$, but it is not the case.

4. Jacobi parameters

Since the measure $\mu_{K(\gamma)}$ is supported on the real line, the polynomials $(Q_n)_{n=0}^\infty$ satisfy a three-term recurrence relation

$$Q_{n+1}(x) = (x - b_{n+1})Q_n(x) - a_n^2\, Q_{n-1}(x), \qquad n \in \mathbb{N}_0.$$

The recurrence starts from $Q_{-1} := 0$ and $Q_0 = 1$. The Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$ define the matrix

$$\begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.1}$$

where $\mu_{K(\gamma)}$ is the spectral measure for the unit vector δ_1 and the self-adjoint operator J on $l_2(\mathbb{N})$, which is defined by this matrix. We are interested in the analysis of asymptotic behavior of $(a_n)_{n=1}^\infty$. Since $\mu_{K(\gamma)}$ is symmetric about $x = 1/2$, all b_n are equal to $1/2$. It is known (see e.g. [30]) that $a_n > 0$, $||Q_n|| = a_1 \cdots a_n$, which, in turn, is the reciprocal to the leading coefficient of the orthonormal polynomial of degree n.

In the next lemmas we use the equality $\int Q_n\, Q_m\, Q_{n+m}\, d\mu_{K(\gamma)} = ||Q_{n+m}||^2$, which follows by orthogonality of Q_{n+m} to all polynomials of smaller degree.

Lemma 4.1. *For all $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$Q_{2^s(2k+1)} = Q_{2^s} \cdot Q_{2^{s+1}k} - \frac{||Q_{2^{s+1}k}||^2}{||Q_{2^s(2k-1)}||^2}\, Q_{2^s(2k-1)}.$$

Proof. Consider the polynomial $P = Q_{2^s} \cdot Q_{2^{s+1}k} - \frac{||Q_{2^{s+1}k}||^2}{||Q_{2^s(2k-1)}||^2} Q_{2^s(2k-1)}$. Since $\deg(Q_{2^s} \cdot Q_{2^{s+1}k}) > \deg Q_{2^s(2k-1)}$, it is a monic polynomial of degree $2^s(2k+1)$. Let us show that P is orthogonal to Q_n for all n with $0 \le n < 2^s(2k+1)$. This will mean that $P = Q_{2^s(2k+1)}$.

Suppose $0 \le n < 2^s(2k-1)$. Then orthogonality follows by comparison of the degrees.

If $n = 2^s(2k-1)$, then $\int P\, Q_n\, d\mu_{K(\gamma)} = 0$ due to the choice of coefficient of the addend in P and the remark above.

Let $2^s(2k-1) < n < 2^s(2k+1)$. We show that $\int Q_{2^s}\, Q_{2^{s+1}k}\, Q_n\, d\mu_{K(\gamma)} = 0$. We write k in the form $k = 2^q(2l+1)$ with some $q, l \in \mathbb{N}_0$. In turn, $n = 2^m(2p+1)$ with $m \neq s$. By Corollary 3.5, $Q_{2^{s+1}k}$ is a linear combination of products of $Q_{2^{s_j}}$ with $\min s_j = s + 1 + q$ in every product. Similarly for Q_n, but here the smallest degree is 2^m. Therefore, $Q_{2^s}\, Q_{2^{s+1}k}\, Q_n$ is a linear combination of $A-$polynomials and for each $A-$polynomial the exponent of the smallest term is 1. By Lemma 3.1(d), the corresponding integral is zero. □

Lemma 4.2. *For all $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$a^2_{2^s(2k+1)}\, a^2_{2^s(2k+1)-1} \cdots a^2_{2^{s+1}k+1} + a^2_{2^{s+1}k}\, a^2_{2^{s+1}k-1} \cdots a^2_{2^{s+1}k-2^s+1} = \|Q_{2^s}\|^2.$$

Proof. By Lemma 4.1 and the remark above,

$$||Q_{2^s(2k+1)}||^2 = \int Q^2_{2^s}\, Q^2_{2^{s+1}k} d\mu_{K(\gamma)} - \frac{\|Q_{2^{s+1}k}\|^4}{\|Q_{2^s(2k-1)}\|^2}. \tag{4.2}$$

Let us show that

$$\int Q^2_{2^s}\, Q^2_{2^{s+1}k} d\mu_{K(\gamma)} = ||Q_{2^s}||^2||Q_{2^{s+1}k}||^2.$$

If $k = 2^m$, we have this immediately, by Lemma 3.1(a).

Otherwise, $2^{s+1}k = 2^m(2l+1)$ with $l \in \mathbb{N}$ and $m \geq s+1$. Then, by Corollary 3.5, $Q_{2^{s+1}k}$ is a linear combination of products $Q_{2^{s_q}} \cdots Q_{2^{s_j}} \cdots Q_{2^m}$ with $s_j > m$ except for the last term. From here, $Q^2_{2^{s+1}k} = Q^2_{2^m} \cdot \sum \alpha_j\, A_j$, where $\sum \alpha_j\, A_j$ is a linear combination of $A-$type polynomials with $s_1 > m$ for each A_j. Therefore,

$$||Q_{2^{s+1}k}||^2 = \sum \alpha_j \int A_j Q^2_{2^m}\, d\mu_{K(\gamma)}.$$

On the other hand,

$$\int Q^2_{2^s}\, Q^2_{2^{s+1}k} d\mu_{K(\gamma)} = \sum \alpha_j \int A_j Q^2_{2^m}\, Q^2_{2^s}\, d\mu_{K(\gamma)}.$$

By Corollary 3.3, this is $||Q_{2^s}||^2||Q_{2^{s+1}k}||^2$.

Therefore, (4.2) can be written as

$$\frac{\|Q_{2^s(2k+1)}\|^2}{\|Q_{2^{s+1}k}\|^2} = \|Q_{2^s}\|^2 - \frac{\|Q_{2^{s+1}k}\|^2}{\|Q_{2^s(2k-1)}\|^2},$$

which is the desired result, as $a_n = \|Q_n\| \,/\, \|Q_{n-1}\|$. □

Theorem 4.3. *The recurrence coefficients $(a_n)_{n=1}^{\infty}$ can be calculated recursively by using Lemma* 4.2 *and* (3.1)*.*

Proof. We already have $a_1 = ||Q_1||$ and $a_2 = ||Q_2|| \,/\, ||Q_1||$. Suppose, by induction, that all a_i are given up to $i = n$. If $n+1 = 2^s > 2$, then

$$a_{n+1} = \frac{||Q_{2^s}||}{||Q_{2^{s-1}}|| \cdot a_{2^{s-1}+1} \cdot a_{2^{s-1}+2} \cdots a_{2^s-1}}, \tag{4.3}$$

where the norms of polynomials can be found by (3.1).

Otherwise, $n+1 = 2^s(2k+1)$ for some $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$. By Lemma 4.2, we have

$$a^2_{n+1} = a^2_{2^s(2k+1)} = \frac{\|Q_{2^s}\|^2 - a^2_{2^{s+1}k} \cdots a^2_{2^{s+1}k-2^s+1}}{a^2_{2^s(2k+1)-1} \cdots a^2_{2^{s+1}k+1}}, \tag{4.4}$$

provided $s \neq 0$. If $s = 0$, then the denominator in the fraction above is absent. This gives a_{n+1}, since the recurrence coefficients are positive. □

In order to illustrate the theorem, let us consider the cases of small s.
If $s = 0$, then $n+1 = 2k+1$ and $a^2_{2k+1} = a^2_1 - a^2_{2k}$. Next, for $s = 1$ and $s = 2$,

$$a^2_{4k+2} = \frac{||Q_2||^2 - a^2_{4k}\, a^2_{4k-1}}{a^2_{4k+1}},\ a^2_{8k+4} = \frac{||Q_4||^2 - a^2_{8k}\, a^2_{8k-1}\, a^2_{8k-2}\, a^2_{8k-3}}{a^2_{8k+3}\, a^2_{8k+2}\, a^2_{8k+1}},\ \text{etc.}$$

Thus, $a_1 = \frac{\sqrt{1-2\gamma_1}}{2}$, $a_2 = \frac{\sqrt{1-2\gamma_2}}{\sqrt{1-2\gamma_1}}\,\gamma_1$, $a_3^2 = a_1^2 - a_2^2$, $a_4 = \frac{\gamma_1\gamma_2\sqrt{1-2\gamma_3}}{a_3\sqrt{1-2\gamma_2}}$, $a_5^2 = a_1^2 - a_4^2$, etc.

Remark 4.4. If $\gamma_n < 1/4$ for $1 \le n \le s$ and $\gamma_n = 1/4$ for $n > s$, then $K(\gamma) = E_s = (2P_{2^s}/r_s + 1)^{-1}[-1,1]$. Here $(P_{2^n} + r_n/2)_{n=0}^{\infty}$ are the Chebyshev polynomials for E_s, as is easy to check. Therefore Theorems 2.8 and 4.3 are applicable for this case as well. For further information about Jacobi parameters corresponding to equilibrium measures of polynomial inverse images, we refer the reader to the article [15].

Remark 4.5. Suppose $\gamma_n = 1/4$ for $n \le N$ with $2^s \le N < 2^{s+1}$. Then $a_1 = 1/\sqrt{8}$ and $a_2 = a_3 = \cdots = a_{2^{s+1}-1} = 1/4$. In particular, if $\gamma_n = 1/4$ for all n, then $a_n = 1/4$ for all $n \ge 2$, which corresponds to the case of the Chebyshev polynomials of the first kind on $[0,1]$.

Lemma 4.6. *Suppose $\gamma_s \le 1/6$ for all s. For fixed $s \in \mathbb{N}_0$, let $c = \frac{4\gamma_{s+1}^2}{(1-2\gamma_{s+1})^2}$ and $C = \frac{2}{1+\sqrt{1-4c}}$. Then the following inequalities hold with $k \in \mathbb{N}_0$:*

(a) If $n = 2^s(2k+1)$, then

$$\frac{1}{2}||Q_{2^s}||^2 \le C^{-1}\,||Q_{2^s}||^2 \le a_n^2 \cdots a_{n-2^s+1}^2 \le ||Q_{2^s}||^2.$$

(b) If $n = 2^s(2k+2)$, then

$$a_n^2 \cdots a_{n-2^s+1}^2 \le C\,\frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2} \le 2\frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2}.$$

Proof. Note that, if γ_{s+1} increases from 0 to 1/6, then c increases from 0 to 1/4 and C increases from 1 to 2. By (3.1) and the definition of r_s, we get

$$\frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2} = \gamma_{s+1}^2\, r_s^2\,\frac{1-2\gamma_{s+2}}{1-2\gamma_{s+1}} = (1-2\gamma_{s+2})\,c\,\|Q_{2^s}\|^2 < \|Q_{2^s}\|^2/4. \tag{4.5}$$

We proceed by induction. For a fixed $s \in \mathbb{N}_0$, let $k = 0$. Then we have at once

$$a_{2^s}^2 \cdots a_1^2 = \|Q_{2^s}\|^2 \quad \text{and} \quad a_{2^{s+1}}^2 \cdots a_{2^s+1}^2 = \frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2}.$$

Suppose (a), (b) are satisfied for $k \le m$. We apply Lemma 4.2 with $k = m+1$:

$$a_{2^s(2m+3)}^2 \cdots a_{2^s(2m+2)+1}^2 + a_{2^s(2m+2)}^2 \cdots a_{2^s(2m+2)-2^s+1}^2 = \|Q_{2^s}\|^2,$$

where for the addend we can use (b) for $k = m$. Therefore,

$$\|Q_{2^s}\|^2 - C\frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2} \le a_{2^s(2m+3)}^2 \cdots a_{2^s(2m+2)+1}^2 \le \|Q_{2^s}\|^2,$$

which is (a) for $k = m+1$, by (4.5).

Next, we claim that

$$a_{2^s(2m+4)}^2 \cdots a_{2^s(2m+2)+1}^2 \le \|Q_{2^{s+1}}\|^2 \tag{4.6}$$

for $m \in \mathbb{N}_0$. If $m = 2l+1$, then we use Lemma 4.2 with $s+1$ instead of s:

$$a_{2^{s+1}(2k+1)}^2 \cdots a_{2^{s+2}k+1}^2 + \text{ positive term } = \|Q_{2^{s+1}}\|^2,$$

which implies (4.6), if we take $k = l+1$, as $2(2k+1) = 2m+4,\ 4k = 2m+2$.

Suppose m is even. Lemma 4.2 now gives

$$\text{positive term } + a^2_{2^s+2k} \cdots a^2_{2^s+2k-2^{s+1}+1} = \|Q_{2^{s+1}}\|^2,$$

where we take $k = m/2+1$. Thus, (4.6) holds true in both cases.

Putting together (a) for $k = m+1$ and (4.6) we get (b) for $k = m+1$. □

Theorem 4.7. *Let $\gamma_s \le 1/6$ for all s. Then $\lim_{s\to\infty} a_{j\cdot 2^s+n} = a_n$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Here, $a_0 := 0$. In particular, $\liminf a_n = 0$.*

Proof. We first show that $\lim_{s\to\infty} a_{j\cdot 2^s} = 0$ for all $j \in \mathbb{N}$. Let $j = 2^l(2k+1)$ where $k, l \in \mathbb{N}_0$. For $s > 0$, the Jacobi parameters admit the following inequality by Lemma 4.6(a):

$$a^2_{2^{s+l}(2k+1)} \cdots a^2_{2^{s+l+1}k+1} \le \|Q_{2^{s+l}}\|^2. \tag{4.7}$$

If $i < s+l$ where $i \in \mathbb{N}_0$, we have $2^{s+l}(2k+1) - 2^i = 2^i(2^{s+l-i}(2k+1)-1)$. Since $2^{s+l-i}(2k+1)-1$ is a positive odd number, by Lemma 4.6(a), we have the inequalities

$$\frac{1}{2}\|Q_{2^i}\|^2 \le a^2_{2^{s+l}(2k+1)-2^i} \cdots a^2_{2^{s+l}(2k+1)-2^{i+1}+1} \quad \text{for} \quad i = 0, \ldots, s+l-1.$$

We multiply these $s+l$ inequalities side by side:

$$2^{-s-l}||Q_1||^2 \cdots ||Q_{2^{s+l-1}}\|^2 \le a^2_{2^{s+l}(2k+1)-1} \cdots a^2_{2^{s+l+1}k+1}$$

and use (4.7):

$$a^2_{j\cdot 2^s} = a^2_{2^{s+l}(2k+1)} \le \frac{2^{s+l}\|Q_{2^{s+l}}\|^2}{\|Q_{2^{s+l-1}}\|^2\|Q_{2^{s+l-2}}\|^2 \cdots \|Q_1\|^2}.$$

By (4.5), the fraction above is bounded by 2^{-s-l+2}. Thus, $\lim_{s\to\infty} a_{j\cdot 2^s} = 0$.

If $n = 1$, then $a^2_{j\cdot 2^s+1} = a_1^2 - a^2_{j\cdot 2^s} \to a_1^2$, which is our claim.

Suppose, by induction, that $\lim_{s\to\infty} a_{j\cdot 2^s+n} = a_n$ for $n = 0, 1, \ldots, m$ and all $j \in \mathbb{N}$. Let $m+1 = 2^p(2q+1)$ where $p, q \in \mathbb{N}_0$. If $q = 0$, then $j\cdot 2^s + m + 1 = j \cdot 2^{s-p} + 1$, so we get the case with $n = 1$. Thus, we can suppose $q \in \mathbb{N}$. Then $j \cdot 2^s + m + 1 = 2^p(2^{s+l-p}(2k+1) + 2q + 1)$ and, for large enough s, we can apply Lemma 4.2:

$$a^2_{j\cdot 2^s+m+1}\, a^2_{j\cdot 2^s+m} \cdots a^2_{j\cdot 2^s+m-2^p+1} + a^2_{j\cdot 2^s+m-2^p} \cdots a^2_{j\cdot 2^s+m+1-2^{p+1}} = \|Q_{2^p}\|^2.$$

Here all indices, except the first, are of the form $j \cdot 2^s + n$ with $n < m+1$. Therefore, by induction hypothesis, $a^2_{j\cdot 2^s+n} \to a_n$ as $s \to \infty$ and

$$(\lim_{s\to\infty} a^2_{j\cdot 2^s+m+1})\, a_m^2 \cdots a^2_{m-2^p+1} + a^2_{m-2^p} \cdots a^2_{m+1-2^{p+1}} = \|Q_{2^p}\|^2.$$

On the other hand, if we apply Lemma 4.2 to the number $m+1$, then we get the same equality with a^2_{m+1} instead of $\lim_{s\to\infty} a^2_{j\cdot 2^s+m+1}$. Since all a_k are positive, we have the desired result. □

Remark 4.8. Since $\liminf a_n = 0$, by [14], $\mu_{K(\gamma)}$ is purely singular. In particular, this implies that $\mu_{K(\gamma)}$ is purely singular continuous since the equilibrium measure cannot have point mass. Moreover, absence of a non-trivial absolutely continuous part of the equilibrium measure, by [24], guarantees that the support has zero Lebesgue measure. Thus $|K(\gamma)| = 0$ if $\gamma_s \le 1/6$ for all $s \in \mathbb{N}$.

5. Widom factors

A finite Borel measure μ supported on a non-polar compact set $K \subset \mathbb{C}$ is said to be regular in the Stahl-Totik sense if $\lim_{n\to\infty} \|Q_n\|^{\frac{1}{n}} = \mathrm{Cap}(K)$ where Q_n is the monic orthogonal polynomial of degree n corresponding to μ. It is known (see, e.g., [28, 31]) that the equilibrium measure is regular in the Stahl-Totik sense. While $\|Q_n\|^{\frac{1}{n}} / \mathrm{Cap}(K)$ has limit 1, the ratio $W_n = \|Q_n\| / (\mathrm{Cap}(K))^n$ may have various asymptotic behavior. We call W_n the *Widom factor* due to the paper [32]. These values play an important role in spectral theory of orthogonal polynomials on several intervals.

Let

$$E = [\alpha, \beta] \setminus \bigcup_{i=1}^{n} (\alpha_i, \beta_i)$$

where $\alpha, \beta \in \mathbb{R}$ and the intervals (α_i, β_i) are disjoint subsets of $[\alpha, \beta]$. Let μ be a unit Borel measure with its support equal to E. Furthermore, let $d\mu(t) = f(t)dt$ on E where f is the Radon-Nikodym derivative of μ with respect to linear Lebesgue measure and $(a_n)_{n=1}^{\infty}$ be the Jacobi parameters corresponding to μ. Then by Theorem 4.1 of [12]

$$\int \log f(t) d\mu_E(t) > -\infty \iff \limsup_{n\to\infty} \frac{a_1 \cdots a_n}{\mathrm{Cap}(E)^n} > 0. \tag{5.1}$$

For further generalizations and different aspects of this result, see [10, 12, 13, 23, 29].

We already know that $\mathrm{Cap}(K(\gamma)) = \exp\left(\sum_{k=1}^{\infty} 2^{-k} \log \gamma_k\right)$. In terms of $(\gamma_k)_{k=1}^{\infty}$ we can rewrite $\|Q_{2^s}\|$ as

$$\frac{\sqrt{1-2\gamma_{s+1}}}{2} \exp\left(2^s \sum_{k=1}^{s} 2^{-k} \log \gamma_k\right). \tag{5.2}$$

Therefore,

$$W_{2^s} = \frac{\sqrt{1-2\gamma_{s+1}}}{2 \exp\left(\sum_{k=s+1}^{\infty} 2^{s-k} \log \gamma_k\right)} \geq \sqrt{2}, \tag{5.3}$$

since $\gamma_s \leq 1/4$. The limit values $\gamma_s = 1/4$ for all s give the Widom factors for the equilibrium measure on $[0, 1]$.

Clearly, (5.3) implies that $\limsup W_n > 0$. If $\gamma_s \leq 1/6$ for all s, then

$$W_{2^s} \geq \sqrt{6}. \tag{5.4}$$

Let us show that, in this case, $\liminf W_n > 0$.

Theorem 5.1. *Let $(W_n)_{n=1}^{\infty}$ be Widom factors for $\mu_{K(\gamma)}$ where $\gamma_s \leq 1/6$ for all s. Then*

(a) $\liminf_{s\to\infty} W_{2^s} = \liminf_{n\to\infty} W_n$.
(b) $\limsup_{n\to\infty} W_n = \infty$.

Proof. (a) We show that $W_n > W_{2^s}$ for $2^s < n < 2^{s+1}$. Let $n = 2^s + 2^{s_1} + \ldots + 2^{s_m}$ with $s > s_1 > s_2 > \ldots > s_m \geq 0$. Then we decompose the product $a_1 \cdots a_n$ into groups

$$(a_1 \cdots a_{2^s}) \cdot (a_{2^s+1} \cdots a_{2^s+2^{s_1}}) \cdots (a_{2^s+\ldots+2^{s_{m-1}}+1} \cdots a_n).$$

For the first group we have $a_1 \cdots a_{2^s} = ||Q_{2^s}||$. For the second group we use Lemma 4.6(a) with $n = 2^s + 2^{s_1}$: $a_{2^s+1} \cdots a_{2^s+2^{s_1}} \geq ||Q_{2^{s_1}}||/\sqrt{2}$. A similar estimation is valid for all other groups. Therefore,

$$\begin{aligned} W_n &= \frac{a_1 \cdots a_{2^s}}{\mathrm{Cap}(K(\gamma))^{2^s}} \, \frac{a_{2^s+1} \cdots a_{2^s+2^{s_1}}}{\mathrm{Cap}(K(\gamma))^{2^{s_1}}} \cdots \frac{a_{2^s+\cdots+2^{s_{m-1}}+1} \cdots a_n}{\mathrm{Cap}(K(\gamma))^{2^{s_m}}} \\ &\geq W_{2^s} W_{2^{s_1}} \cdots W_{2^{s_m}} (\sqrt{2})^{-m}, \end{aligned}$$

which exceeds $W_{2^s}(\sqrt{3})^m$, by (5.4). From here, $\min_{2^s \leq n < 2^{s+1}} W_n = W_{2^s}$ and the result follows.

(b) Applying the procedure above to W_{2^s-1} and taking the limit gives the desired result. □

In order to illustrate the behavior of Widom factors, let us consider some examples. Suppose $\gamma_s \leq 1/6$ for all s.

Example 5.2. If $\gamma_s \to 0$, then $W_n \to \infty$. Indeed, $W_{2^s} \geq \frac{1}{\sqrt{6}} \exp(\frac{1}{2} \log \frac{1}{\gamma_{s+1}})$.

Example 5.3. There exists $\gamma_s \nrightarrow 0$ with $W_n \to \infty$. Indeed, we can take $\gamma_{2k} = 1/6$, $\gamma_{2k-1} = 1/k$.

Example 5.4. If $\gamma_s \geq c > 0$ for all s, then $\liminf_{n\to\infty} W_n \leq 1/2c$.

Example 5.5. There exists γ with $\inf \gamma_s = 0$ and $\liminf_{n\to\infty} W_n < \infty$. Here we can take $\gamma_s = 1/6$ for $s \neq s_k$ and $\gamma_{s_k} = 1/k$ for a sparse sequence $(s_k)_{k=1}^\infty$. Then $(W_{2^{s_k}})_{k=1}^\infty$ is bounded.

6. Towards the Szegő class

The convergence of the integral on the left-hand side of (5.1) defines the Szegő class of spectral measures for the finite gap Jacobi matrices. The Widom condition on the right-hand side is the main candidate to characterize the Szegő class for the general case; see [10, 23, 29].

For the definition of regularity for the Dirichlet problem, see e.g., Chapter 4 in [25]. The equilibrium measure is the most natural measure in the theory of orthogonal polynomials. In particular, for known examples, the values $\limsup W_n$ associated with equilibrium measures are bounded below by positive numbers. So we make the following conjecture:

Conjecture 6.1. *If a compact set K is regular with respect to the Dirichlet problem, then the Widom condition $W_n \nrightarrow 0$ is valid for the equilibrium measure μ_K.*

Concerning the Szegő condition, one can conjecture that the left-hand side of (5.1) can be written as

$$I(\mu) := \int \log(d\mu/d\mu_K) d\mu_K(t) > -\infty \tag{6.1}$$

provided that the support of μ is a perfect non-polar compact set. Indeed, for the finite gap case, this coincides with the condition in (5.1), since the integral $\int \log(d\mu_K/dt) d\mu_K(t)$ converges. By Jensen's inequality (see also Section 4 in [12]), the value $I(\mu)$ is non-positive and it attains its maximum 0 just in the case $\mu = \mu_K$ a.e. with respect to μ_K. On the other hand, there are strong objections to (6.1), based on the numerical evidence from [18], where, for the Cantor-Lebesgue measure μ_{CL} on the classical Cantor set K_0, the Jacobi parameters (a_n) were calculated for

$n \leq 200.000$. Krüger-Simon conjectured that (see e.g. Conjecture 3.2 in [18]) the Widom factors for the Cantor-Lebesgue measure is bounded below by a positive number. Therefore, if we wish to preserve the Widom characterization of the Szegő class, the integral $I(\mu_{CL})$ must converge, but, since μ_{CL} and μ_{K_0} are mutually singular, it is not the case.

References

[1] Gökalp Alpan and Alexander Goncharov, *Two measures on Cantor sets*, J. Approx. Theory **186** (2014), 28–32, DOI 10.1016/j.jat.2014.07.003. MR3251460

[2] G. Alpan and A. Goncharov, *Orthogonal polynomials on generalized Julia sets*, Preprint (2015), arXiv:1503.07098v3

[3] Artur Avila and Svetlana Jitomirskaya, *The Ten Martini Problem*, Ann. of Math. (2) **170** (2009), no. 1, 303–342, DOI 10.4007/annals.2009.170.303. MR2521117 (2011a:47081)

[4] M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, *Orthogonal polynomials associated with invariant measures on Julia sets*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 381–384, DOI 10.1090/S0273-0979-1982-15043-1. MR663789 (84a:42031)

[5] M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, *Infinite-dimensional Jacobi matrices associated with Julia sets*, Proc. Amer. Math. Soc. **88** (1983), no. 4, 625–630, DOI 10.2307/2045451. MR702288 (85a:30040)

[6] M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, *Almost periodic Jacobi matrices associated with Julia sets for polynomials*, Comm. Math. Phys. **99** (1985), no. 3, 303–317. MR795106 (87k:58123)

[7] J. Bellissard, D. Bessis, and P. Moussa, *Chaotic States of Almost Periodic Schrödinger Operators*, Phys. Rev. Lett. **49**, 701–704 (1982)

[8] D. Bessis and P. Moussa, *Orthogonality properties of iterated polynomial mappings*, Comm. Math. Phys. **88** (1983), no. 4, 503–529. MR702566 (85a:58053)

[9] Rainer Brück and Matthias Büger, *Generalized iteration*, Comput. Methods Funct. Theory **3** (2003), no. 1-2, 201–252, DOI 10.1007/BF03321035. MR2082015 (2005f:37088)

[10] Jacob S. Christiansen, *Szegő's theorem on Parreau-Widom sets*, Adv. Math. **229** (2012), no. 2, 1180–1204, DOI 10.1016/j.aim.2011.09.012. MR2855090

[11] Jacob S. Christiansen, Barry Simon, and Maxim Zinchenko, *Finite gap Jacobi matrices, I. The isospectral torus*, Constr. Approx. **32** (2010), no. 1, 1–65, DOI 10.1007/s00365-009-9057-z. MR2659747 (2011h:42035)

[12] Jacob S. Christiansen, Barry Simon, and Maxim Zinchenko, *Finite gap Jacobi matrices, II. The Szegő class*, Constr. Approx. **33** (2011), no. 3, 365–403, DOI 10.1007/s00365-010-9094-7. MR2784484

[13] Jacob S. Christiansen, Barry Simon, and Maxim Zinchenko, *Finite gap Jacobi matrices: a review*, Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy's 60th birthday, Proc. Sympos. Pure Math., vol. 87, Amer. Math. Soc., Providence, RI, 2013, pp. 87–103. MR3087900

[14] J. Dombrowski, *Quasitriangular matrices*, Proc. Amer. Math. Soc. **69** (1978), no. 1, 95–96. MR0467373 (57 #7232)

[15] J. S. Geronimo and W. Van Assche, *Orthogonal polynomials on several intervals via a polynomial mapping*, Trans. Amer. Math. Soc. **308** (1988), no. 2, 559–581, DOI 10.2307/2001092. MR951620 (89f:42021)

[16] Alexander P. Goncharov, *Weakly equilibrium Cantor-type sets*, Potential Anal. **40** (2014), no. 2, 143–161, DOI 10.1007/s11118-013-9344-y. MR3152159

[17] Steven M. Heilman, Philip Owrutsky, and Robert S. Strichartz, *Orthogonal polynomials with respect to self-similar measures*, Exp. Math. **20** (2011), no. 3, 238–259, DOI 10.1080/10586458.2011.564966. MR2836250

[18] H. Krüger and B. Simon, *Cantor polynomials and some related classes of OPRL*, J. Approx. Theory (2014), http://dx.doi.org/10.1016/j.jat.2014.04.003

[19] N. G. Makarov and A. Volberg, *On the harmonic measure of discontinuous fractals*, LOMI preprint, E-6-86, Steklov Mathematical Institute, Leningrad (1986)

[20] G. Mantica, *A stable Stieltjes technique for computing orthogonal polynomials and Jacobi matrices associated with a class of singular measures*, Constr. Approx. **12** (1996), no. 4, 509–530, DOI 10.1007/s003659900028. MR1412197 (97k:33011)

[21] G. Mantica, *Quantum Intermittency in Almost-Periodic Lattice Systems derived from their Spectral Properties*, Physica D, **103**, 576–589 (1997)

[22] G. Mantica, *Numerical computation of the isospectral torus of finite gap sets and of IFS Cantor sets*, Preprint (2015), arXiv:1503.03801

[23] Franz Peherstorfer and Peter Yuditskii, *Asymptotic behavior of polynomials orthonormal on a homogeneous set*, J. Anal. Math. **89** (2003), 113–154, DOI 10.1007/BF02893078. MR1981915 (2004j:42022)

[24] Alexei Poltoratski and Christian Remling, *Reflectionless Herglotz functions and Jacobi matrices*, Comm. Math. Phys. **288** (2009), no. 3, 1007–1021, DOI 10.1007/s00220-008-0696-x. MR2504863 (2010i:47063)

[25] Thomas Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995. MR1334766 (96e:31001)

[26] Edward B. Saff and Vilmos Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 316, Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom. MR1485778 (99h:31001)

[27] Barry Simon, *Almost periodic Schrödinger operators: a review*, Adv. in Appl. Math. **3** (1982), no. 4, 463–490, DOI 10.1016/S0196-8858(82)80018-3. MR682631 (85d:34030)

[28] Barry Simon, *Equilibrium measures and capacities in spectral theory*, Inverse Probl. Imaging **1** (2007), no. 4, 713–772, DOI 10.3934/ipi.2007.1.713. MR2350223 (2008k:31003)

[29] Mikhail Sodin and Peter Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, J. Geom. Anal. **7** (1997), no. 3, 387–435, DOI 10.1007/BF02921627. MR1674798 (2000k:47033)

[30] Vilmos Totik, *Orthogonal polynomials*, Surv. Approx. Theory **1** (2005), 70–125. MR2221567 (2007a:42055)

[31] Harold Widom, *Polynomials associated with measures in the complex plane*, J. Math. Mech. **16** (1967), 997–1013. MR0209448 (35 #346)

[32] Harold Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Advances in Math. **3** (1969), 127–232 (1969). MR0239059 (39 #418)

Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
E-mail address: gokalp@fen.bilkent.edu.tr

Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
E-mail address: goncha@fen.bilkent.edu.tr